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Mid-sem Examination
DA241 (2023)
Solutions and Marking Scheme.

Solution 1. Assume that the doors are labeled such that you choose Door 1.

Let us define the following events:

S : You get the car following the strategy "stick to your original choice".

C_j : Car is behind the door j.

By law of total probability, we condition on which door has the car:

$$P(S) = P(S|C_1) \cdot P(C_1) + \cdots + P(S|C_7) \cdot P(C_7)$$

$$= \frac{1}{7} \quad \text{because:}$$

The car can be behind any door so $P(C_j) = \frac{1}{7} \; \forall j = 1, \ldots, 7$.

Let $M_{i,j,k}$: Monty opens door $i, j,$ and k.

$$P(S) = \sum_{i,j,k} P(S|M_{ijk}) \, P(M_{ijk}). \qquad 2 \le i < j < k \le 7.$$

By symmetry, $P(S|M_{ijk}) = P(S) = \frac{1}{7}$.

$\Rightarrow$ conditional probability that the car is behind one of the remaining 3 doors is $\frac{6}{7}$, which implies you should switch because then your success probability is $\frac{2}{7}$ rather than $\frac{1}{7}$ if you stick with your initial choice.

Solution 2. We are given that $X \sim Exponential(\lambda)$. $\lambda \geq 0$

$$\therefore f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & o/w. \end{cases}$$

We need to find $E(X^3)$ using the MGF of X.

$$M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}$$

$$= \left. \frac{\lambda e^{-(\lambda-t)x}}{-(\lambda - t)} \right|_0^\infty$$

$$= \lambda \left(0 + \frac{1}{\lambda - t} \right) \quad \lambda > t.$$

$$= \frac{\lambda}{\lambda - t} \quad t < \lambda.$$

Now we need to use this MGF to find $E(X^3)$.

One way: We can take the third derivative of $M(t)$ & evaluate it at $t = 0$.

$$M(t) = \frac{\lambda}{\lambda - t} \quad t < \lambda$$

$$= \frac{1}{1 - \left(\frac{t}{\lambda}\right)} = \left(1 - \frac{t}{\lambda}\right)^{-1}.$$

$$\frac{\partial M(t)}{\partial t} = \frac{1}{\lambda} \left(1 - \frac{t}{\lambda}\right)^{-2}$$

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$$\frac{\partial^2 M(t)}{\partial t^2} = \frac{2}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-3}$$

$$\frac{\partial^3 M(t)}{\partial t^3} = \frac{6}{\lambda^3}\left(1-\frac{t}{\lambda}\right)^{-4}$$

$$\Rightarrow E(X^3) = \left.\frac{\partial^3 M(t)}{\partial t^3}\right|_{t=0} = \frac{6}{\lambda^3}$$

Second way: Pattern recognition:

$$M(t) = \frac{1}{1-\frac{t}{\lambda}} = \sum_{n=0}^{\infty}\left(\frac{t}{\lambda}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{n!\, t^n}{\lambda^n\, n!} \quad \text{for } t < \lambda.$$

The nth moment of X is the coeffecient of $\frac{t^n}{n!}$ in the

$$\therefore E(X^3) = \frac{3!}{\lambda^3} = \frac{6}{\lambda^3}.$$

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Solution 3.

X : # of times the code fails before the first successful run.

We recognize that X is a geometric random variable with probability p of success. — 1 mark

$$\therefore P(X=k) = q^k p \qquad k = 0,1,2,\ldots$$

$$E(X) = \sum_{k=0}^{\infty} k q^k p.$$

$$= p \sum_{k=0}^{\infty} k q^k. \qquad (*)$$

We can use the trick from the lecture:

We know, $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$.

Taking derivative on both sides, we have

$$\sum_{k=1}^{\infty} k q^{k-1} = \frac{+1}{(1-q)^2} \qquad -(**).$$

$$\Rightarrow \sum_{k=1}^{\infty} k q^k = \frac{q}{(1-q)^2}.$$

Using the above in $(*)$, we have:

$$E(X) = \frac{p \cdot q}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}. \qquad \text{— 1 mark}$$

classmate

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$Var(X) = E(X^2) - (E(X))^2$.

$E(X^2) = p \sum_{k=0}^{\infty} k^2 q^k$

$= p \sum_{k=0}^{\infty} k(k-1) q^k + p \sum_{k=0}^{\infty} k q^k$.

Using the same trick as before:

$\sum_{k=2}^{\infty} k(k-1) q^{k-2} = \frac{2}{(1-q)^3}$, (taking derivative of ~~(...)~~).

—

$\Rightarrow E(X^2) = \frac{2pq^2}{p^3} + E(X)$

$= \frac{2q^2}{p^2} + E(X) = \frac{2q^2}{p^2} + \frac{q}{p}$.

— 2 marks.

$Var(X) \quad \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$

$= \frac{q^2}{p^2} + \frac{q}{p} = \frac{q^2 + qp}{p^2} = \frac{q(p+q)}{p^2}$

$= \frac{q}{p^2}$.

— 1 mark.

Solution 1.

Let A_i : ith card in the deck has the number i written on it.

We want to find

$P(A_1 \cup A_2 \cup .. \cup A_n)$ because we win if one of the cards has a matching number to its position.

To find the probability of the union of A_i's, we will use inclusion -exclusion.

First, $P(A_i) = \frac{1}{n}$ $\{ P(A_i) = \frac{(n-1)!}{n!}$ - Fix ith card in ith position, rest $n-1$ can be anywhere $\}$.

Second, $P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$ (Fix ith & jth position of cards numbered i & j)

Similarly, $P(A_i \cap A_j \cap A_k) = \frac{1}{n(n-1)(n-2)}$ following the same pattern.

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Inclusion-exclusion formula:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

$$\therefore P\left(\bigcup_{i=1}^{n} A_i\right) = n \cdot \frac{1}{n} - \binom{n}{2} \frac{1}{n(n-1)} + \binom{n}{3} \frac{1}{n(n-1)(n-2)} -$$

$$\cdots + (-1)^{n+1} \frac{1}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1} \frac{1}{n!}$$

For large n, this probability is close to $1 - e^{-1} = 0.63$.

A random point (X, Y, Z) is chosen uniformly in ball B

$$B = \{(x, y, z) : x^2+y^2+z^2 \leq 1\}.$$

a) The probability in three dimensions will be proportional to volume, same as it is proportional to area in two dimensions & length in one-dimension.

$$P((X,Y,Z) \in A) = c \cdot \text{Volume}(A). \quad A \subseteq B.$$

where c is a constant.

if $A = B$, $P((X,Y,Z) \in B) = c$ (uniform),

and $\frac{1}{c} = \frac{4}{3}\pi$, volume of the ball:

$$\therefore f(x,y,z) = \begin{cases} \frac{3}{4\pi} & \text{if } x^2+y^2+z^2 \leq 1 \\ 0 & \text{o/w.} \end{cases}$$

b) Joint PDF of X and Y,

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f(x,y,z)\, dz$$

(integrate out Z from the joint PDF of X,Y,Z).

Since $x^2+y^2+z^2 \leq 1$

$$\Rightarrow z^2 \leq 1 - x^2 - y^2$$

$$-\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}.$$

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$$\therefore f_{X,Y}(x,y) = \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{3}{4\pi}\, dz.$$

$$= \frac{3}{4\pi}\, 2 \cdot \int_{0}^{\sqrt{1-x^2-y^2}} 1\, dz$$

$$= \begin{cases} \frac{3}{2\pi}\sqrt{1-x^2-y^2} & \text{if } x^2+y^2 \le 1. \\ 0 & \text{o/w}. \end{cases}$$

c). From part b), we can integrate out y from $f_{X,Y}(x,y)$ to get marginal of X.

$$f_X(x) = \frac{3}{2\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy.$$

$$-1 \le x \le 1.$$